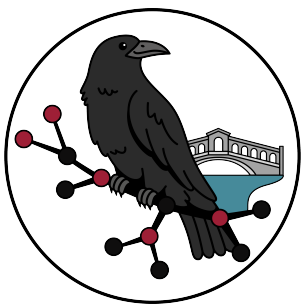
Dynamic Programming

CPSCP – Collaborative Problem Solving and Competitive Programming

Ruben Becker



RAVEN – Research on Algorithms Venice

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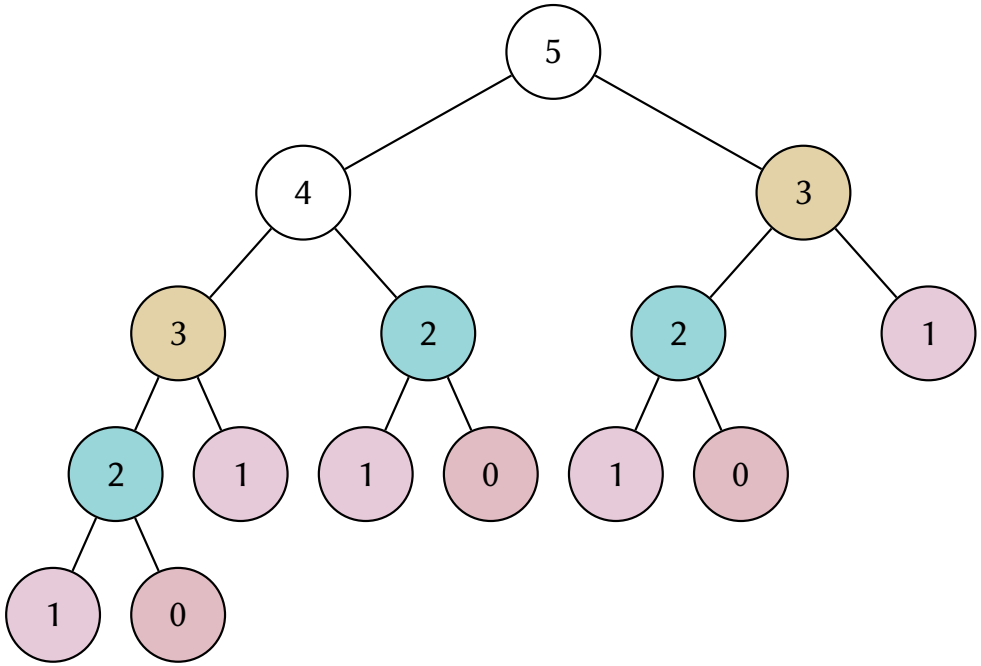

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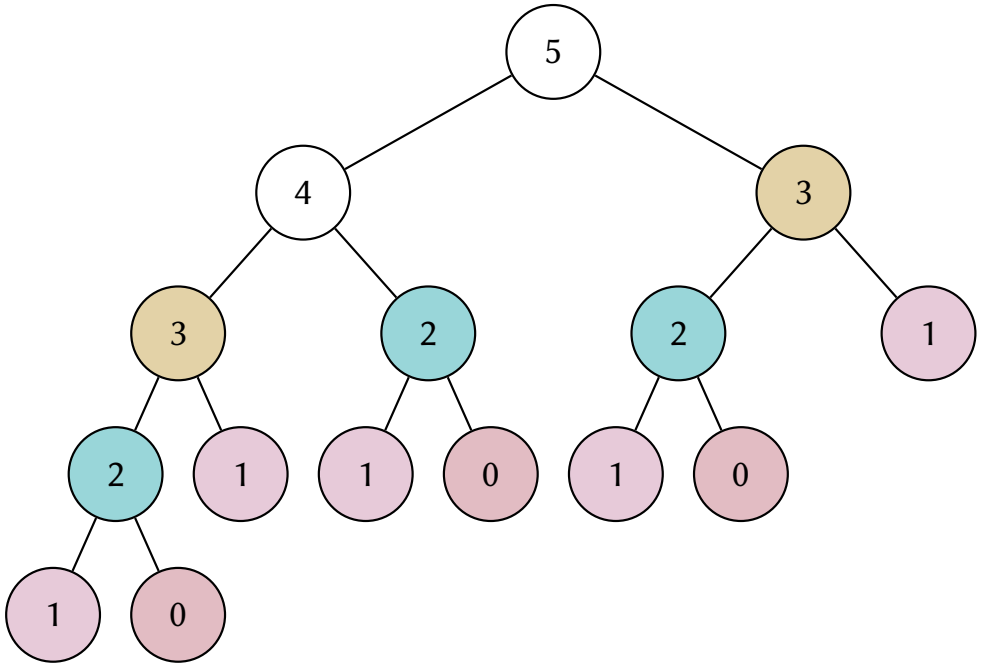

same-colored nodes = same argument, recomputed

Dynamic Programming

Key observation

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$\Rightarrow$ each value computed **once**

$\Rightarrow$ $O(n)$ time, $O(n)$ space

Top-Down vs. Bottom-Up

Memoization (top-down)

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- both $O(n)$ time
- tabulation avoids recursion overhead and can drop to $O(1)$ space

Another Example: Longest Common Subsequence

Problem: length of the LCS of s_1 (len. n), s_2 (len. m).

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$s_1 =$ **A**B**CD**, $s_2 =$ **AC**B**D**

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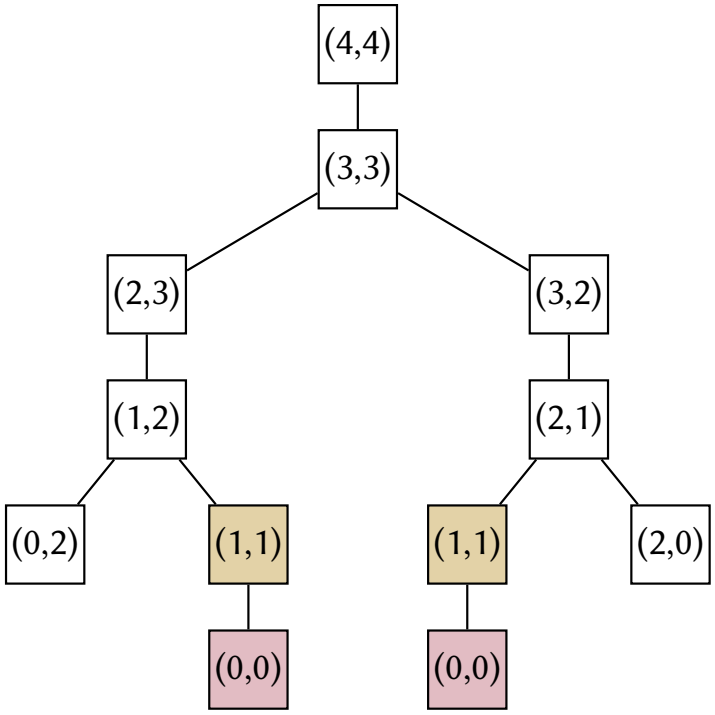

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		A	C	B	D
	0	0	0	0	0
A	0	1	1	1	1
B	0	1	1	2	2
C	0	1	2	2	2
D	0	1	2	2	**3**

dp filled bottom-up: no state recomputed

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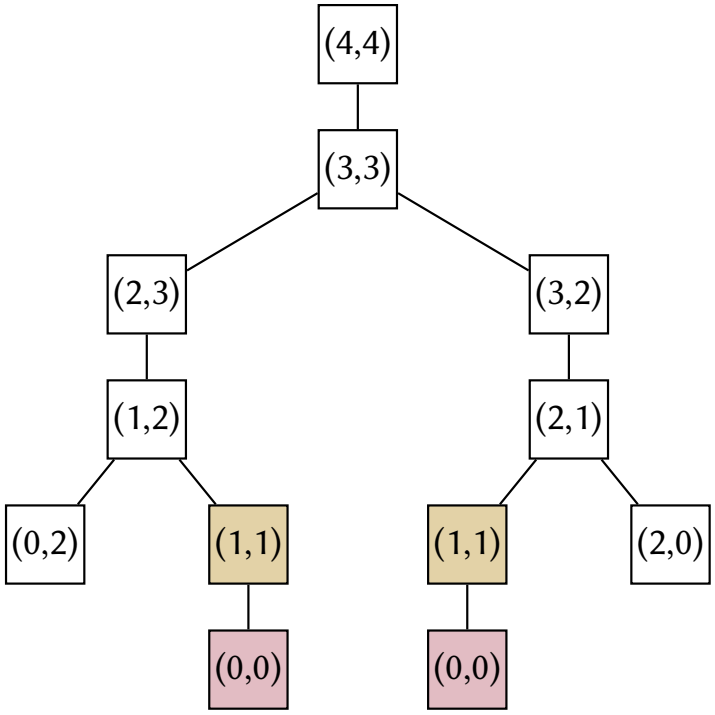

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naive	$O(\varphi^n)$	$O(2^{n+m})$
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Recipe

1. **State**: Which parameters describe subproblem?
   E.g., number/pair of indices
2. **Recurrence**: write how one state results from previous states
3. **Overlap?** Yes $\rightarrow$ DP.
   No $\rightarrow$ Simple Recursion.
4. **Store & Reuse**: Either by memoization (top-down) or Tabulation (bottom-up)

Your Task

- https://open.kattis.com/problems/knapsack
- https://open.kattis.com/problems/trainsorting